③ Наћи локалне екстремне вриједности:

$$f(x,y,z) = 2x^2 - xy + 2xz - y + y^3 + z^2$$

Рјешење:

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 4x - y + 2z = 0 \\ \frac{\partial f}{\partial y} &= -x - 1 + 3y^2 = 0 \\ \frac{\partial f}{\partial z} &= 2x + 2z = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} &y = 4x + 2z = 2x \\ &y^2 = \frac{x+1}{3} \\ &z = -x \end{aligned}$$

$$(2x)^2 = \frac{x+1}{3}$$

$$4x^2 = \frac{x+1}{3} \Rightarrow 12x^2 - x - 1 = 0$$

$$x_{1,2} = \frac{1 \pm 7}{24}$$

$$x_1 = \frac{8}{24};\ x_2 = -\frac{6}{24}$$

$$\underbrace{x_1 = \frac{1}{3};\ x_2 = -\frac{1}{4},}$$

Стационарне тачке: $A\left(\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right)$, $B\left(-\frac{1}{4}, -\frac{1}{2}, \frac{1}{4}\right)$

Нађимо друге парцијалне изводе.

$$\frac{\partial^2 f}{\partial x^2} = 4,\quad \frac{\partial^2 f}{\partial x \partial y} = -1,\quad \frac{\partial^2 f}{\partial x \partial z} = 2;\quad \frac{\partial^2 f}{\partial y^2} = 6y,\quad \frac{\partial^2 f}{\partial y \partial z} = 0,\quad \frac{\partial^2 f}{\partial z^2} = 2$$

I) У тачки A,

$$\begin{pmatrix} 4 & -1 & 2 \\ -1 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$\Delta_1 = 4 > 0$$

$$\Delta_2 = \begin{vmatrix} 4 & -1 \\ -1 & 4 \end{vmatrix} = 16 - 1 = 15 > 0$$

$$\Delta_3 = \begin{vmatrix} 4 & -1 & 2 \\ -1 & 4 & 0 \\ 2 & 0 & 2 \end{vmatrix} = 2\cdot(-8) + 2\cdot 15 = 14 > 0$$

У тачки је локални минимум, $f_{min}(A) = f\left(\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right) =$
$= 2\cdot\left(\frac{1}{3}\right)^2 - \frac{1}{3}\cdot\frac{2}{3} + 2\cdot\frac{1}{3}\left(-\frac{1}{3}\right) - \frac{2}{3} + \left(\frac{2}{3}\right)^3 + \left(-\frac{1}{3}\right)^2 = \ldots$

У тачки B,

$$\begin{pmatrix} 4 & -1 & 2 \\ -1 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$\Delta_1 = 4 > 0$$

$$\Delta_2 = \begin{vmatrix} 4 & -1 \\ -1 & -3 \end{vmatrix} = -12 - 1 = -13 < 0$$

$$\Delta_3 = \begin{vmatrix} 4 & -1 & 2 \\ -1 & -3 & 0 \\ 2 & 0 & 2 \end{vmatrix} = 2\cdot 6 + 2\cdot(-13) < 0$$

Нема екстремума у B.

<u>Други начин:</u>

$d^2f(A) = ?$

$$d^2f = \frac{\partial^2 f}{\partial x^2}dx^2 + 2\cdot\frac{\partial^2 f}{\partial x \partial y}dxdy + 2\cdot\frac{\partial^2 f}{\partial x \partial z}\cdot dxdz + \frac{\partial^2 f}{\partial y^2}dy^2 + 2\cdot\frac{\partial^2 f}{\partial y \partial z}dydz + \frac{\partial^2 f}{\partial z^2}dz^2$$

$$\begin{aligned} d^2f(A) &= 4dx^2 - 2dxdy + 4dxdz + 4dy^2 + 2dz^2 \\ &= 2(2dx^2 - dxdy + 2dxdz + 2dy^2 + dz^2) \\ &= 2(dx^2 + 2dxdz + dz^2 + dx^2 - dxdy + dy^2 + dy^2) \\ &= 2((dx + dz)^2 + (dx - \tfrac{1}{2}dy)^2 - \tfrac{1}{4}dy^2 + dy^2 + dy^2) \end{aligned}$$

Одавде,

$d^2f(A)>0$ за све $dx^2+dy^2+dz^2\neq 0$

$d^2f(A)=0$ за $\left.\begin{array}{l} dx+dz=0 \\ dx-\frac{1}{2}dy=0 \\ dy=0 \end{array}\right\} \Rightarrow dx=dy=dz=0$

Дакле, тачка A је локални минимум.

$d^2f(B)=?$

$d^2f(B)=4dx^2-2dxdy+4dxdz-3dy^2+2dz^2$

Ако узмемо $dy=dz=0$, а $dx\neq 0$ $\quad d^2f(B)=4dx^2>0$

" – $\quad dx=dz=0$, а $dy\neq 0$ $\quad d^2f(B)=-3dy^2<0$.

па $d^2f(B)$ мијења знак $\Rightarrow$ Нема екстремума у B.

4 Наћи локалне екстремуме ф-је:

$$f(x,y,z)=x^2+y^2+z^2+(4-x-y-z)^2$$ (домаћи)

Рјешење: $A(1,1,1)$ је локални минимум.

$f_{min}(A)=4.$

Условни екстремуми

1. Одредити екстремуме ф-је $f(x,y,z)=2x+y-2z$ при услову $x^2+y^2+z^2=36$.

Решење:

$$F(x,y,z,\lambda)=2x+y-2z+\lambda\cdot(x^2+y^2+z^2-36)$$

$$\left.\begin{aligned} \frac{\partial F}{\partial x}&=2+2\lambda x=0\\ \frac{\partial F}{\partial y}&=1+2\lambda y=0\\ \frac{\partial F}{\partial z}&=-2+2\lambda z=0\\ x^2+y^2&+z^2=36 \end{aligned}\right\} \quad \begin{aligned} 1+\lambda x&=0 &\Rightarrow x&=-\tfrac{1}{\lambda}\\ 1+2\lambda y&=0 & y&=-\tfrac{1}{2\lambda}\\ -1+\lambda z&=0 & z&=\tfrac{1}{\lambda}\\ x^2+y^2+z^2&=36 \end{aligned} \qquad \lambda\neq 0,$$

Ако увратимо x, y, z у ј-ну $x^2+y^2+z^2=36$

$$\frac{1}{\lambda^2}+\frac{1}{4\lambda^2}+\frac{1}{\lambda^2}=36$$

$$\frac{4+1+4}{4\lambda^2}=36 \Rightarrow 4\lambda^2=\frac{9}{36}\,{}_4 \Rightarrow \lambda^2=\frac{1}{16}\Rightarrow \lambda=\pm\frac{1}{4}$$

Стационарне тачке:

$S_1(-4,-2,4)$, $\lambda_1=\frac{1}{4}$

$S_2(4,2,-4)$, $\lambda_2=-\frac{1}{4}$

$$\frac{\partial^2F}{\partial x^2}=2\lambda \quad \frac{\partial^2F}{\partial x\partial y}=0, \quad \frac{\partial^2F}{\partial x\partial z}=0, \quad \frac{\partial^2F}{\partial y^2}=2\lambda, \quad \frac{\partial^2F}{\partial y\partial z}=0, \quad \frac{\partial^2F}{\partial z^2}=2\lambda$$

За S_1,

$$d^2F(S_1)=2\cdot\tfrac{1}{4}dx^2+2\cdot\tfrac{1}{4}dy^2+2\cdot\tfrac{1}{4}dz^2=\tfrac{1}{2}(dx^2+dy^2+dz^2)>0$$

(за $dx^2+dy^2+dz^2\neq 0$)

За S_2, $d^2F(S_2)=-\frac{1}{2}(dx^2+dy^2+dz^2)<0$ (за $dx^2+dy^2+dz^2\neq 0$

Ф-ја f на $x^2+y^2+z^2=36$ у тачки S_1 има локални миним[ум] а у S_2 локални максимум.

$f_{min}(S_1)=-18$, $f_{max}(S_2)=18$

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(1)

(Završni ispit) - priprema

3,4) Zadaci iz dvostrukog i trostrukog integrala kao se materijala sa sajta koji su objavljeni u toku semestra.

5) Riješiti dif. j-ne:

a) $y'' - 3y' + 2y = e^x(x+2)$

Rj.

$y'' - 3y' + 2y = 0$

$k^2 - 3k + 2 = 0$

$k_{1,2} = \frac{3 \pm \sqrt{9-8}}{2}$

$k_{1,2} = \frac{3 \pm 1}{2}$

$k_1 = 2, \quad k_2 = 1$

$y_h = C_1 e^{2x} + C_2 e^x$

$f(x) = e^{1\cdot x}(x+2) \Rightarrow \lambda = 1$

$x+2 \to P_1(x)$,

$y_p = x \cdot e^x \cdot (ax+b) = e^x(ax^2+bx)$

$y_p' = e^x(ax^2+bx) + e^x(2ax+b) =$

$= e^x(ax^2 + bx + 2ax + b)$

$y_p'' = e^x(ax^2+bx+2ax+b) + e^x(2ax+b+2a)$

$y_p'' - 3y_p' + 2y_p = e^x(x+2)$

$e^x(ax^2+bx+2ax+b+2ax+b+2a) - 3e^x(ax^2+bx+2ax+b) +$

$+ 2e^x(ax^2+bx) = e^x(x+2) \quad / : e^x$

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$\cancel{ax^2} + \cancel{bx} + 4ax + 2b + 2a - \cancel{3ax^2} - 3bx - 6ax - 3b + \cancel{2ax^2} + 2bx =$

$= x + 2$

$-2ax - b + 2a = x + 2$

$-2a = 1 \Rightarrow a = -\frac{1}{2}$

$2a - b = 2$

$2 \cdot \left(-\frac{1}{2}\right) - b = 2$

$\underline{b = -3}$

$\underline{y_p = xe^x \cdot \left(-\frac{1}{2}x - 3\right)}$

$\underline{y = y_h + y_p}$

b) $y'' - 4y = e^{2x}$ c) $y'' - 7y' + 6y = (x-2)e^x$

d) $y'' + 4y = \cos 2x$

$f(x) = \cos 2x = e^{0 \cdot x}(1 \cdot \cos 2x + 0 \sin 2x) \Rightarrow$

$k^2 + 4 = 0$

$k^2 = -4$

$\underline{k = \pm 2i}$

$y_p = x(a\cos 2x + b \sin 2x)$

$y_p' = a\cos 2x + b\sin 2x + x(-2a\sin 2x + 2b\cos 2x)$

$\underline{y_h = C_1 \cos 2x + C_2 \sin 2x}$

$y_p'' = -4x(a\cos 2x - b\sin 2x) + 4(-a\sin 2x + b\cos 2x).$

$-4x(-a\cos 2x - b\sin 2x) + 4(b\cos 2x - a\sin 2x) + 4x(a\cos 2x + b\sin 2x) = \cos 2x$

$-4a = 0 \Rightarrow a = 0$

$4b = 1 \Rightarrow b = \frac{1}{4}.$

$y_p = \frac{1}{4}x\sin 2x.$ $y = y_h + y_p$

e) Odrediti ekstremne vrij. fje $u = x^2+y^2-12x+16y$ uz uslov $x^2+y^2=25$.

f) Odrediti ekstremne vrij. fje $f(x,y) = -2x+3y-1$ uz uslov $4x^2+y^2 = 40$.

3) Riješiti diferencijalne j-ne:

a) $xy' + 4y = x^4y^2$, $y(2) = -1$

b) $y' = \dfrac{2xy}{x^2-y^2}$; $y(1) = 1$

c) $y' + 4y = 2xy^2$

d) $xy^2\,dy = (x^3+y^3)\,dx$

e) $y'' - 16y = (7-x^2)\sin 4x$

f) $y'' + y' - 2y = (x^2-1)e^{2x}$

g) $y'' + 2y' - 3y = 2xe^{-3x}$